Journal of Statistical Physics, Vol. 50, Nos. 5/6, 1988

Critical Properties of Rule 22 Elementary Cellular Automata

John G. Zabolitzky[1]

Received December 1, 1987

The one-dimensional elementary cellular automaton "Rule 22" is studied by means of Monte Carlo simulation on the dedicated K2 high-speed computer. If one considers random initialization with probability p for "one"-initialization per site, it is shown that the system behaves like a normal one-dimensional statistical ensemble with critical points at $p = 0$ and $p = 1$. Critical slowing down is exhibited, with a dynamical exponent of 1.0. The standard initialization of $p = 0.5$ is too far away from the critical point to allow similar observations.

KEY WORDS: Cellular automata; Monte Carlo simulation; special-purpose computer; critical properties.

1. INTRODUCTION

One-dimensional cellular automata have recently been studied intensely.[1-5] Some nontrivial statistical properties have been exhibited.[6] In particular, the so-called "Rule 22" automaton was found to be the most interesting one. However, it has not been possible to extract critical exponents from the data measured so far. It is the purpose of the present paper to establish a firmer correspondence between the behavior of one-dimensional cellular automata and other statistical ensembles in one dimension, as discussed, e.g., by Stauffer (Ref. 7, Appendix), taken here as a reference for the "normal" behavior of one-dimensional statistical systems.

The "Rule 22" automaton is constituted by a circular chain of N sites, where at each site a single bit of data is located, i.e., a binary variable with value equal to either zero or unity. We introduce a variable parameter

[1] KONTRON Elektronik, 8057 Eching, West-Germany, and Minnesota Supercomputer Institute and School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota.

0022-4715/88/0300-1255$06.00/0

$0 \leqslant p \leqslant 1$, which specifies the initialization of the system at time $t = 0$. The parameter p gives the probability for each site independently to assume the value one; otherwise, the site assumes the value zero. Time evolution is given by "Rule 22": whenever the sum of a site and its two direct neighbors is equal to one, that site assumes a value of one in the next time step; otherwise, it assumes zero. The updating process is carried out simultaneously for all N sites at each step in discrete time. Grassberger[6] found highly unusual behavior for $p = 0.5$ in Rule 22.

As a paradigm, I use the analogy to a magnetic system, like the Q2R cellular automaton approximating an Ising spin system.[8,9] In that case the initialization probability plays a role like the temperature in real magnetic systems, i.e., there exists a critical probability p_c which exhibits all the properties of a critical parameter. I will show that the same holds true for the one-dimensional automata studied here. However, the critical point is at $p = 0$, as is normal for one-dimensional systems.[7] The point $p = 1$ is similar to $p = 0$, since, after one iteration beginning from $p = 1 - \varepsilon$, one obtains a state similar to the $p = \varepsilon^2$ initialization (probability of finding two zeros out of three sites).

2. SIMULATIONAL TECHNIQUES

In order to avoid the necessity to compute finite-size corrections in the statistical analysis of the Monte Carlo data taken, I use a large system of $N = 64{,}000$ sites, stored in 4000 16-bit machine words of the K2 special-purpose microprogrammed array processor.[10] With a system size as large as this, finite-size corrections do not appear to the accuracy of the data obtained in the present simulations. As a function of time (number of steps taken), I compute the average number of sites with value one,

$$m(t) = \sum_i s_i(t)/N, \qquad 0 \leqslant m \leqslant 1$$

On the K2 processor the update algorithm, including the bit count operation for the computation of the "magnetization" m, occurs at a rate of 30 million updates per second. Initial configurations are set according to the specified probability p by drawing a random number r, $0 \leqslant r \leqslant 1$, and setting the site to one if and only if $r \leqslant p$. It is extremely critical that the random number generator used does not exhibit any correlations and be of good statistical quality. Otherwise, the system may easily lock into persistent metastable states, resulting in rather inaccurate results for finite computational time. I used the generator RAN1 of Ref. 11 after experiencing severe problems with a number of other generators.

For each probability p, from 400 to 1500 independent simulations were carried out and averaged over, each one run for 30,000 time steps. The total CPU time used therefore is on the order of several hundred hours.

3. RESULTS AND DISCUSSION

A typical time evolution is shown in Fig. 1 for the case of $p = 0.02$, averaged over 750 samples. The decadic logarithm of the absolute deviation of the magnetization $m(t)$ from the final steady-state equilibrium value, $m(\infty) = 0.35096$,[6] is shown. We observe a region of approximately exponential decay, followed by statistical fluctuations. The rapid oscillations of this value within the exponential regime are *not* statistical, but are a reproducible consequence of the Rule 22 time evolution procedure, as was already evident from the analytical results of Ref. 6. The statistical errors of the magnetizations given are of the order of the statistical fluctuations within the equilibrium regime. The equilibrium

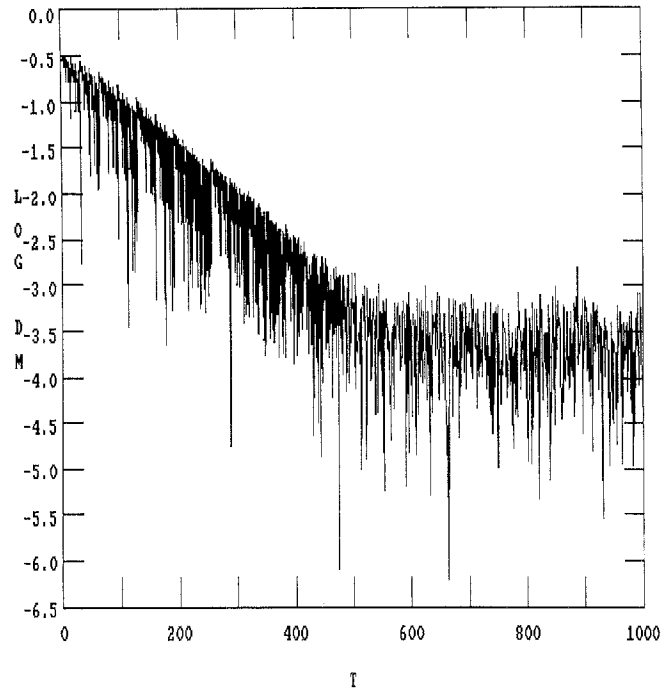

Fig. 1. Deviation of time-dependent magnetization $m(t)$ from equilibrium value $m(\infty)$, decadic logarithm of absolute value, for $p = 0.02$ initialization, as a function of time.

magnetization $m(\infty)$ does not depend upon the initialization probability p and coincides with the one found in Ref. 6 for all p calculated.

In order to extract more quantitative information, Fig. 2 shows a gliding average of the same data,

$$m_{\mathrm{av}}(t_i) = \sum_{j=i-2}^{j=i+2} m(t_j)/5$$

The oscillations are significantly reduced, and extraction of the decay constant is possible, assuming the formal ansatz

$$|m(t) - m(\infty)| \propto e^{-at}$$

depicted as the straight line. Because of the residual fluctuations, it is quite evident that higher order terms cannot be extracted. Consequently, the extracted decay constant is associated with a rather large possible error. The very same procedure can be carried out for any arbitrary initialization probability p chosen, resulting in Table I. It is seen clearly that for small p,

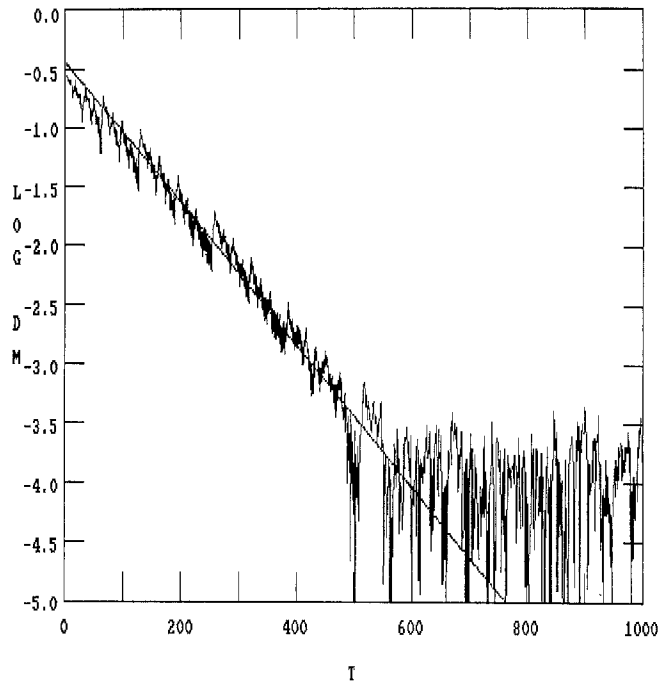

Fig. 2. Same as Fig. 1, but with gliding average over five time steps. The straight line is the best exponential fit to the data in the range $t = 100$–500.

Table I. Decay Constants *a* Extracted for Various Initialization Probabilities *p*

p	*a*	*p*	*a*
0.0002	0.000026(8)	0.1	0.055(6)
0.0005	0.00018(5)	0.2	0.077(8)
0.001	0.00025(5)	0.3	0.14(2)
0.002	0.0008(2)	0.4	0.13(2)
0.005	0.0015(3)	0.5	0.13(2)
0.01	0.0074(8)	0.6	0.16(2)
0.02	0.013(2)	0.7	0.086(9)
0.05	0.034(4)	0.8	0.054(6)
		0.9	0.019(2)

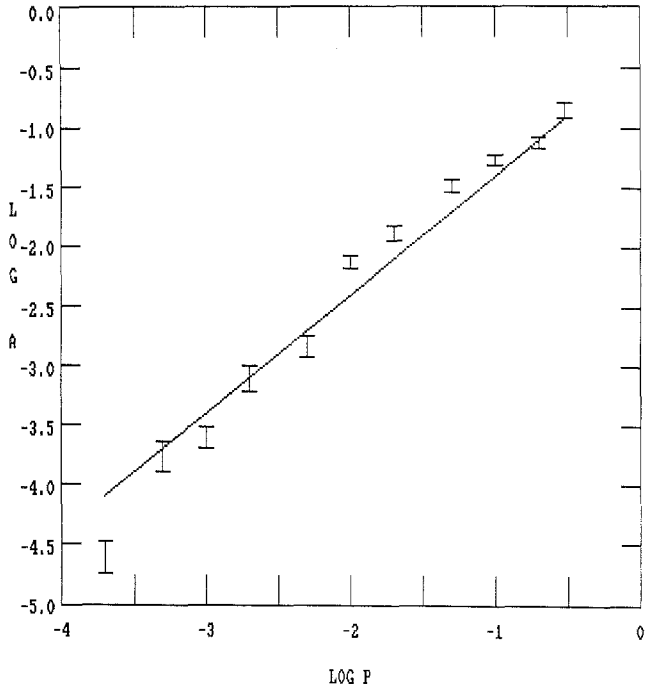

Fig. 3. Decadic logarithm of decay constant a as a function of decadic logarithm of initialization probability p. The straight line indicates the best fit; see text.

the decay constant is linear in p. This result—as well as the individual decay constants themselves—does not depend upon the width of the gliding average taken to the accuracy studied here.

Previous studies[6] limited themselves to the special case $p = 0.5$. It is quite obvious now that the very fast decay at this value of p, coupled with the significant oscillations, does not allow one to extract meaningful quantitative information. Essential information is restricted to the small-p regime. Approaching $p = 0$, the decay becomes arbitrarily slow, indicating the presence of the critical point $p = 0$.

Figure 3 shows a log–log plot of the data in Table I. It is seen that the behavior approaching $p = 0$ is indeed linear (slope unity), with a coefficient of 0.44(5), i.e., dynamical critical exponent of unity,

$$a(p) = 0.44(5)p$$

4. STRUCTURE OF EQUILIBRIUM STATE

Characteristic structural elements of the equilibrium state are rather long runs of zeros .[6] In analogy to percolation theory,[7] I define a cluster size distribution

$$n_s = \text{average number of clusters [runs] of } s \text{ zeros}$$

per cluster site, with the sum rule (normalization condition)

$$\sum_s s n_s + m(\infty) = 1$$

For independent sites, i.e., uncorrelated (percolation) system of the same magnetization as the rule 22 equilibrium state, this cluster size distribution follows[7]

$$n_s \propto [1 - m(\infty)]^s = 0.64904^s$$

The average cluster distribution obtained from 1340 independent samples of the $N = 30{,}000$ system in equilibrium (after more than 10,000 time steps, starting from $p = 0.5$) is shown in Fig. 4. The rather accurate best fit to the data for large s is

$$n_s \propto 0.75(3)^s$$

i.e., there are many more large clusters of zeros due to correlation effects than in the uncorrelated system. I suggest that it might be possible to find a

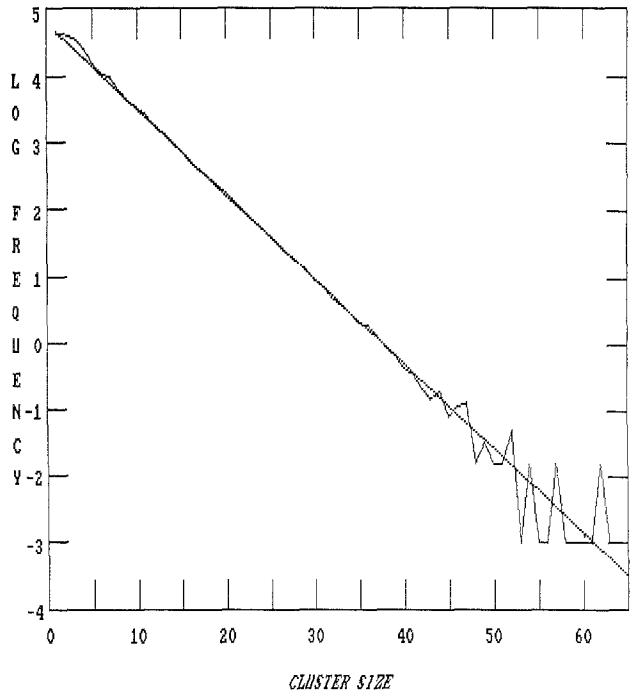

Fig. 4. Cluster size distribution for runs of zeros in equilibrium state, natural logarithm of cluster frequency versus cluster size.

proof to the conjecture that this constant is equal to 3/4, indicating some yet hidden simplicity in the structure of the equilibrium state of the Rule 22 automaton.

ACKNOWLEDGMENTS

I am grateful to P. Grassberger for directing my attention to the present problem, as well as to D. Stauffer for elucidating discussions.

REFERENCES

1. S. Wolfram, *Rev. Mod. Phys.* **55**:601 (1983).
2. P. Grassberger, *Phys. Rev. A* **28**:3666 (1983).
3. N. H. Packard and S. Wolfram, *J. Stat. Phys.* **38**:901 (1985).
4. P. Grassberger, *Physica D* **10**:52 (1984); *Int. J. Theor. Phys.* **25**:907 (1986); *J. Stat. Phys.* **38**:901 (1985).
5. S. Wolfram, *Theory and Applications of Cellular Automata* (World Scientific, Singapore, 1986).

6. P. Grassberger, *J. Stat. Phys.* **45**:27 (1986); preprint (1987).
7. D. Stauffer, *Phys. Rep.* **54**:1 (1979).
8. G. Vichniac, *Physica D* **10**:96 (1986); Y. Pomeau, *J. Phys. A* **17**:L415 (1984).
9. H. J. Herrmann, *J. Stat. Phys.* **45**:145 (1986); J. G. Zabolitzky and H. J. Herrmann, *J. Comp. Phys.* (1988).
10. J. Deckert, S. Wansleben, and J. G. Zabolitzky, *Phys. Rev. D* **35**:683 (1987).
11. W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes* (Cambridge, 1986).